6.3 Q.3

$|f(x)| \leq |x^2 \sin(\frac{1}{x})| \leq x^2$ and $\lim_{x\to 0} x^2 = 0$

By Sandwich Thm, $\lim_{x\to 0} f(x) = 0$.

Also, $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 = 0$.

$$\frac{f(x)}{g(x)} = \sin\frac{1}{x}.$$

<u>Claim</u>: $\lim_{x\to 0} \sin\frac{1}{x}$ DNE.

<u>Proof</u>: Let $a_n = \frac{1}{\pi n}$ & $b_n = \frac{1}{\pi(2n+\frac{1}{2})}$. Note that $a_n, b_n \neq 0$ and $a_n, b_n \to 0$ as $n\to\infty$. However,

$$\lim_{n\to\infty} \sin\frac{1}{a_n} = \lim_{n\to\infty} \sin(\pi n) = 0$$

$$\lim_{n\to\infty} \sin\frac{1}{b_n} = \lim_{n\to\infty} \sin(\pi(2n+\tfrac{1}{2})) = 1 \neq 0$$

$\therefore \lim_{x\to 0} \sin\frac{1}{x}$ DNE by the sequential criterion. $\square$

6.3 Q.14

(The equality holds only when $c>0$, $c\neq\frac{1}{e}$. The question is missing this assumption.)

When $c>0$, $c\neq\frac{1}{e}$, by L'Hôpital's Rule,

$$\lim_{x\to c}\frac{x^c-c^x}{x^x-c^c}=\lim_{x\to c}\frac{e^{c\ln x}-e^{x\ln c}}{e^{x\ln x}-c^c}$$

$$=\lim_{x\to c}\frac{\frac{c}{x}e^{c\ln x}-\ln c\cdot e^{x\ln c}}{(\ln x+1)\,e^{x\ln x}}$$

$$=\frac{e^{c\ln c}-\ln c\cdot e^{c\ln c}}{(\ln c+1)\,e^{c\ln c}}$$

$$=\frac{1-\ln c}{\ln c+1}$$

6.4 Q.7

Let $f(x) = (1+x)^{1/3}$, $x > -1$.

$f'(x) = \frac{1}{3}(1+x)^{-2/3}$

$f''(x) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\cdot(1+x)^{-5/3} = -\frac{2}{9}(1+x)^{-5/3}$

$f(0) = 1$, $f'(0) = \frac{1}{3}$, $\frac{f''(0)}{2!} = -\frac{1}{9}$

$\therefore$ $1+\frac{1}{3}x-\frac{1}{9}x^2$ is the 2^{nd} order Taylor's polynomial of f at 0.

$f^{(3)}(x) = \left(-\frac{2}{9}\right)\cdot\left(-\frac{5}{3}\right)(1+x)^{-8/3} = \frac{10}{27}(1+x)^{-8/3}$

By the Taylor's remainder thm, there exists some c between 0 & x

s.t. $(1+x)^{1/3} - \left(1+\frac{1}{3}x-\frac{1}{9}x^2\right) = \frac{f^{(3)}(c)}{3!}x^3$

$$= \frac{5}{81}(1+c)^{-8/3}x^3$$

When $x>0$, $c>0$.

$$\therefore \left|(1+x)^{1/3} - \left(1+\frac{1}{3}x-\frac{1}{9}x^2\right)\right| = \left|\frac{5}{81}(1+c)^{-8/3}x^3\right|$$

$$\leq \frac{5}{81}x^3$$

When $x = 0.2$,

$$\sqrt[3]{1.2} \approx 1+\frac{1}{3}\times 0.2 - \frac{1}{9}\times 0.2^2 = \frac{239}{225}$$

$$\text{Error} \leq \frac{5}{81}(0.2^3) \approx 4.94\times 10^{-4}$$

When $x = 1$,

$$\sqrt[3]{2} \approx 1 + \frac{1}{3} - \frac{1}{9} = \frac{11}{9}$$

$$\text{Error} \le \frac{5}{81}$$

6.4 Q.8

$f^{(n)}(x) = e^x$

For each fixed x, x_0, the remainder term is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}$$

$$= \frac{e^{c_n}(x-x_0)^{n+1}}{(n+1)!}$$ for some c_n between x & x_0.

WLOG, $x \neq x_0$.

By thm 3.2.11, it suffices to show that $L := \lim_{n\to\infty} \frac{|R_{n+1}(x)|}{|R_n(x)|}$ exists & $L<1$.

$$\left|\frac{R_{n+1}(x)}{R_n(x)}\right| = \left|\frac{\frac{e^{c_n}(x-x_0)^{n+2}}{(n+2)!}}{\frac{e^{c_{n+1}}(x-x_0)^{n+1}}{(n+1)!}}\right| \le e^{|x-x_0|}\frac{|x-x_0|}{n+2}$$

Note that $\lim_{n\to\infty} e^{|x-x_0|} \cdot \frac{|x-x_0|}{n+2} = 0$

By Sandwich thm,

$$\therefore L = \lim_{n\to\infty} \frac{|R_{n+1}(x)|}{|R_n(x)|} = 0 < 1$$

$\Rightarrow \lim_{x\to\infty} R_n(x) = 0$ by thm 3.2.11.

6.4 Q.12

Let $f(x) = \sin x,\ x \in [0,1]$ $\Rightarrow f(0) = 0$

$f'(x) = \cos x$ $\Rightarrow f'(0) = 1$

$f''(x) = -\sin x$ $\Rightarrow \frac{f''(0)}{2!} = 0$

$f^{(3)}(x) = -\cos x$ $\Rightarrow \frac{f^{(3)}(0)}{3!} = -\frac{1}{6}$

$f^{(4)}(x) = \sin x$ $\Rightarrow \frac{f^{(4)}(0)}{4!} = 0$

$f^{(5)}(x) = \cos x$ $\Rightarrow \frac{f^{(5)}(0)}{5!} = \frac{1}{120}$

$f^{(6)}(x) = -\sin x$ $\Rightarrow \frac{f^{(6)}(0)}{6!} = 0$

$\therefore\ x - \frac{x^3}{6} + \frac{x^5}{120}$ is the 6^{th} order Taylor's polynomial at 0.

$f^{(7)}(x) = -\cos x$

For each x, $\exists\, c$ between 0 & x s.t.

$$\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) = \frac{f^{(7)}(c)}{7!} x^7 = \frac{-\cos c}{5040} x^7$$

If $x = 0$, the estimate is trivial.

If $x \neq 0$, $c \neq 0$. Then $|\cos c| < 1$.

$$\therefore \left|\sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right| = \left|\frac{-\cos c}{5040} x^7\right| < \frac{1}{5040}$$

6.4 Q.16

By def.,

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h}$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. when $0 < |h| < \delta$, $\left| \frac{f'(a+h) - f'(a)}{h} - f''(a) \right| < \varepsilon$.

Replacing h by $-h$, we get $\left| \frac{f'(a) - f'(a-h)}{h} - f''(a) \right| < \varepsilon$.

$$\therefore \lim_{h\to 0} \frac{f'(a) - f'(a-h)}{h} = f''(a)$$

By L'Hôpital's Rule,

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

$$= \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} \quad (\text{L'Hôpital})$$

$$= \lim_{h\to 0} \frac{1}{2} \left(\frac{f'(a+h) - f'(a) + f'(a) - f'(a-h)}{h} \right)$$

$$= \frac{1}{2}(f''(a) + f''(a))$$

$$= f''(a)$$

Let $f(x) = x|x|$, $x \in (-1, 1)$, $a = 0$. Rewrite

$$f(x) = \begin{cases} x^2, & x \in [0,1) \\ -x^2, & x \in (-1, 0) \end{cases}$$

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 - 0}{h} = 0$$

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{-h^2 - 0}{h} = 0$$

$\therefore f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ exists and is equal to 0.

$$\lim_{h\to 0^+} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0^+} \frac{2h-0}{h} = 2$$

$$\lim_{h\to 0^-} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0^-} \frac{-2h-0}{h} = -2 \neq 2$$

Then $f''(0) = \lim_{h\to 0} \frac{f'(h)-f'(0)}{h}$ DNE.

However,

$$\lim_{h\to 0} \frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

$$= \lim_{h\to 0} \frac{h|h|-h|h|}{h^2}$$

$$= 0$$